Problem solving and search

Chapter 3

Artificial Intelligence

Slides from AIMA — `http://aima.cs.berkeley.edu`

Outline

- Problem-solving agents
- Problem types
- Problem formulation
- Example problems
- Basic search algorithms
- Informed search algorithms

Problem-solving agents

Restricted form of general agent:

```
function SIMPLE-PROBLEM-SOLVING-AGENT( percept) returns an action
    static: seq, an action sequence, initially empty
            state, some description of the current world state
            goal, a goal, initially null
            problem, a problem formulation

    state ← UPDATE-STATE(state, percept)
    if seq is empty then
        goal ← FORMULATE-GOAL(state)
        problem ← FORMULATE-PROBLEM(state, goal)
        seq ← SEARCH( problem)
    action ← RECOMMENDATION(seq, state)
    seq ← REMAINDER(seq, state)
    return action
```

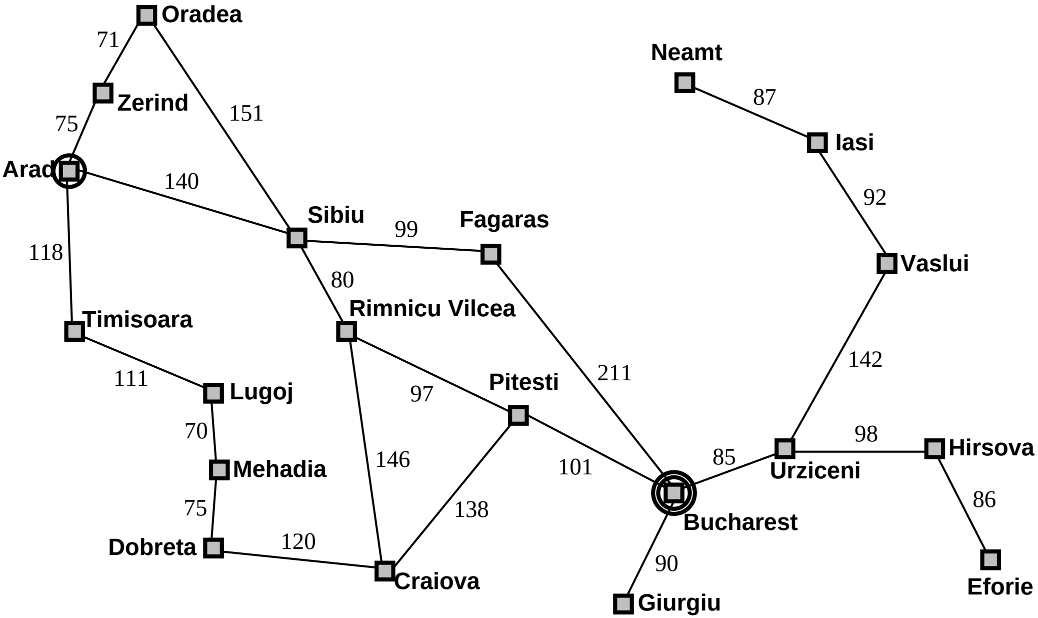
Note: this is offline problem solving; solution executed "eyes closed."
Online problem solving involves acting without complete knowledge.

Example: Romania

On holiday in Romania; currently in Arad.
Flight leaves tomorrow from Bucharest

- Formulate goal: be in Bucharest
- states: various cities
- actions: drive between cities
- Find solution: sequence of cities, e.g., Arad, Sibiu, Fagaras, Bucharest

Example: Romania



Problem types

- Deterministic, fully observable $\Longrightarrow$ single-state problem
  - Agent knows exactly which state it will be in; solution is a sequence
- Non-observable $\Longrightarrow$ conformant problem
  - Agent may have no idea where it is; solution (if any) is a sequence
- Nondeterministic and/or partially observable $\Longrightarrow$ contingency problem
  - percepts provide **new** information about current state
  - solution is a contingent plan or a policy
  - often **interleave** search, execution
- Unknown state space $\Longrightarrow$ exploration problem ("online")

Example: vacuum world

Single-state, start in #5. Solution??

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Contingency, start in #5
Murphy's Law: *Suck* can dirty a clean carpet
Local sensing: dirt, location only.
Solution??

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Local sensing: dirt, location only.
Solution??
[*Right*, **if** *dirt* **then** *Suck*]

Single-state problem formulation

A problem is defined by four items:

- initial state e.g., "at Arad"
- successor function $S(x)$ = set of action–state pairs
  e.g., $S(Arad) = \{\langle Arad \rightarrow Zerind, Zerind \rangle, \ldots\}$
- goal test, can be
  - explicit, e.g., x = "at Bucharest"
  - implicit, e.g., $NoDirt(x)$
- path cost (additive) e.g., sum of distances, number of actions executed, etc. $c(x, a, y)$ is the step cost, assumed to be ≥ 0

A solution is a sequence of actions leading from the initial state to a goal state

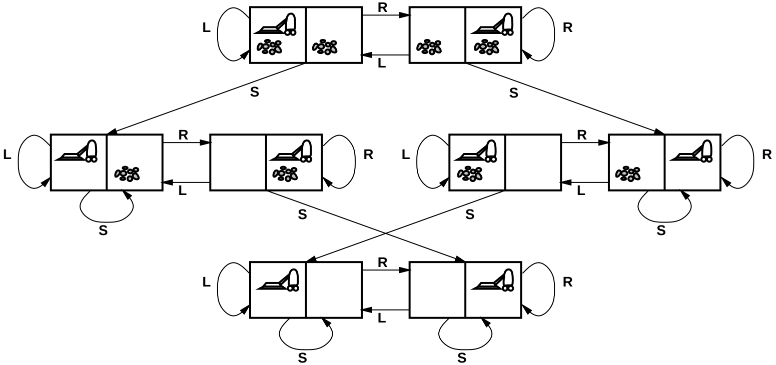
Selecting a state space

Real world is absurdly complex
$\Rightarrow$ state space must be **abstracted** for problem solving

- (Abstract) state = set of real states
- (Abstract) action = complex combination of real actions
  e.g., "Arad $\rightarrow$ Zerind" represents a complex set of possible routes, detours, rest stops, etc.
  For guaranteed realizability, **any** real state "in Arad" must get to some real state "in Zerind"
- (Abstract) solution = set of real paths that are solutions in the real world

Each abstract action should be "easier" than the original problem!

Example: vacuum world state space graph



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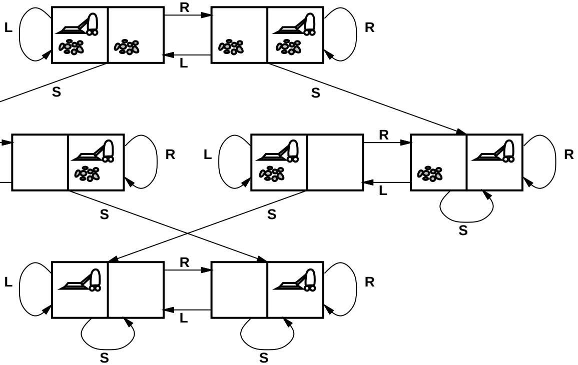

- states??: integer dirt and robot locations (ignore dirt amounts etc.)
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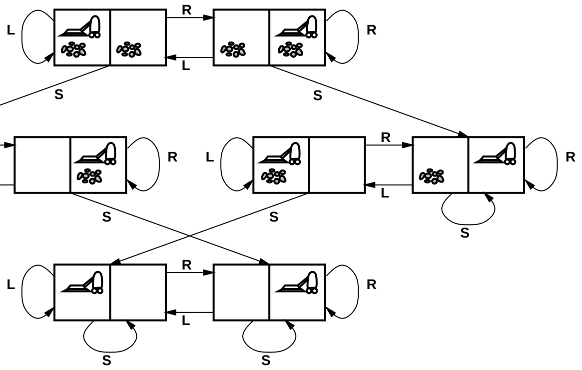

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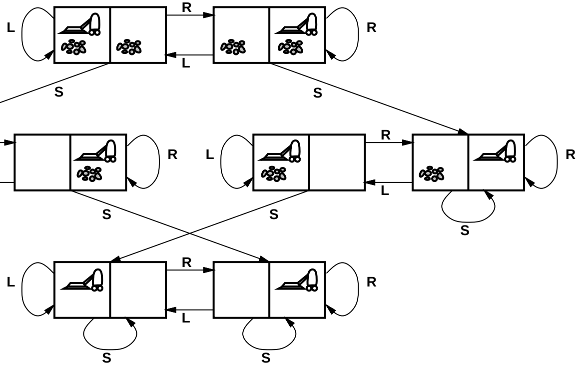

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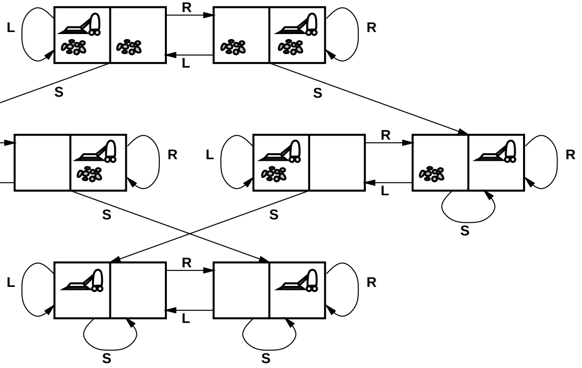
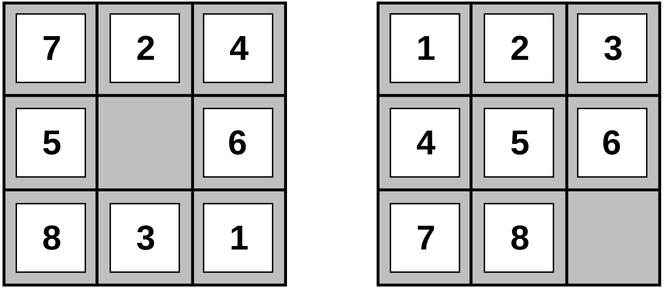


- states??: integer dirt and robot locations (ignore dirt amounts etc.)
- actions??: *Left*, *Right*, *Suck*, *NoOp*
- goal test??: no dirt
- path cost??: 1 per action (0 for *NoOp*)

Example: The 8-puzzle



Start State **Goal State**

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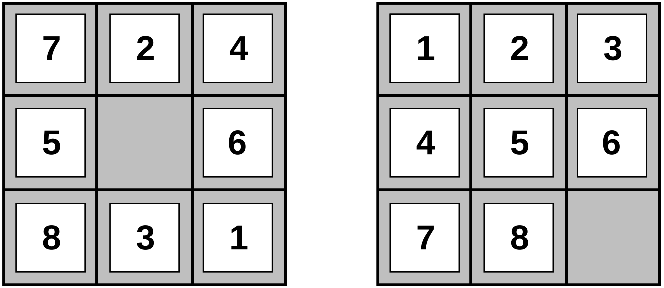

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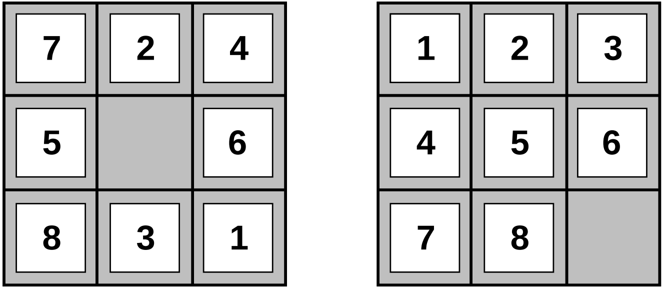

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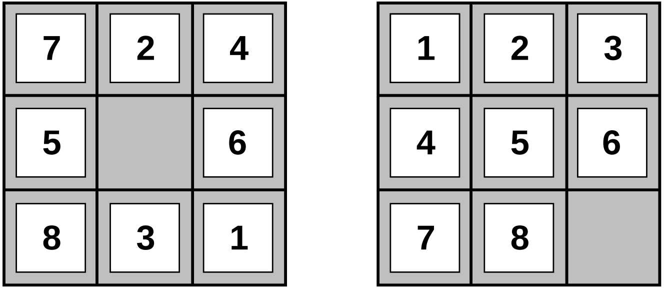

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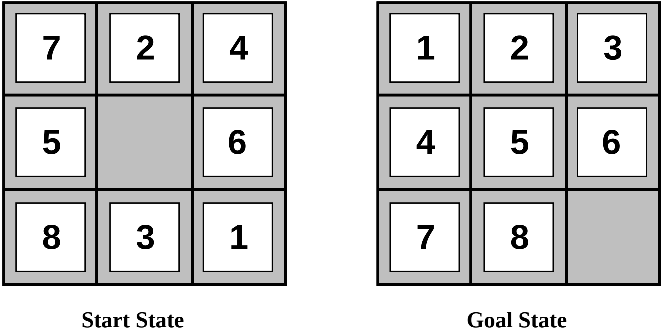
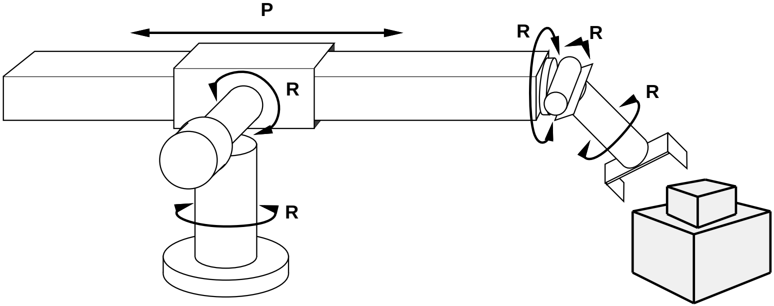


- states??: integer locations of tiles (ignore intermediate positions)
- actions??: move blank left, right, up, down (ignore unjamming etc.)
- goal test??: = goal state (given)
- path cost??: 1 per move

[Note: optimal solution of *n*-Puzzle family is NP-hard]

Example: robotic assembly



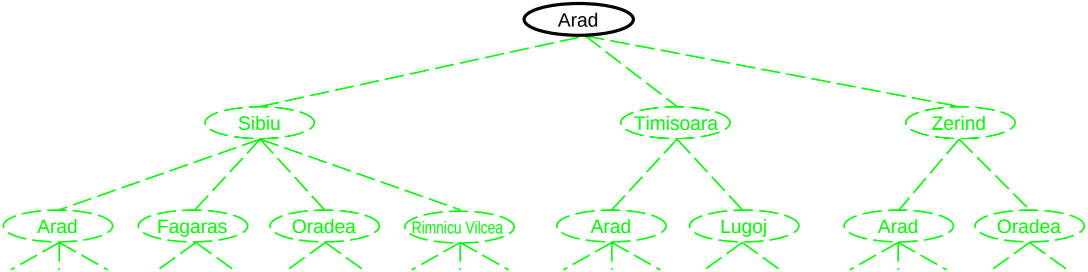
- states??: real-valued coordinates of robot joint angles
  parts of the object to be assembled
- actions??: continuous motions of robot joints
- goal test??: complete assembly **with no robot included!**
- path cost??: time to execute

Tree search algorithms

Basic idea: offline, simulated exploration of state space by generating successors of already-explored states (a.k.a. expanding states)

```
function TREE-SEARCH( problem, strategy) returns a solution, or failure
    initialize the search tree using the initial state of problem
    loop do
        if there are no candidates for expansion then return failure
        choose a leaf node for expansion according to strategy
        if the node contains a goal state then return the corresponding solution
        else expand the node and add the resulting nodes to the search tree
    end
```

Tree search example



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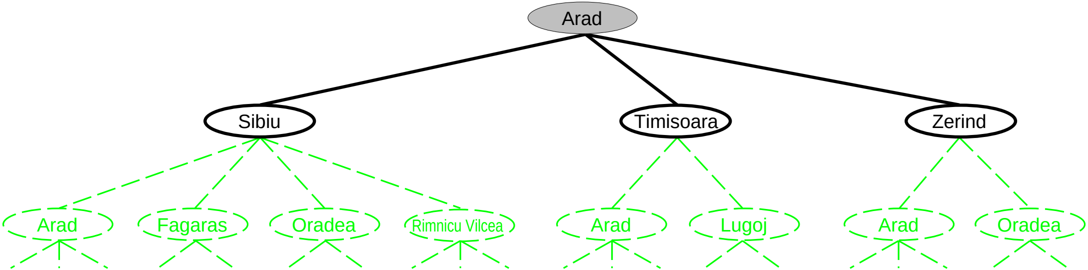

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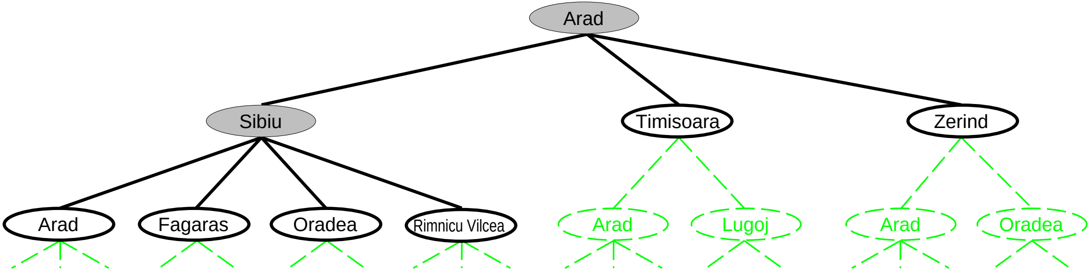
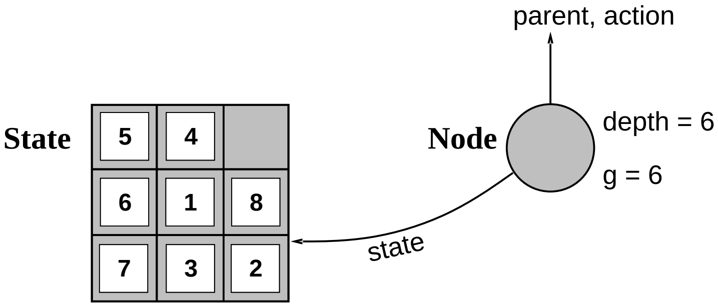


Implementation: states vs. nodes

- A state is a (representation of) a physical configuration
- A node is a data structure constituting part of a search tree
  includes parent, children, depth, path cost $g(x)$

States do not have parents, children, depth, or path cost!



The Expand function creates new nodes, filling in the various fields and using the SuccessorFn of the problem to create the corresponding states

Implementation: general tree search

```
function TREE-SEARCH( problem, fringe) returns a solution, or failure
    fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
    loop do
        if fringe is empty then return failure
        node ← REMOVE-FRONT(fringe)
        if GOAL-TEST(problem, STATE(node)) then return node
        fringe ← INSERTALL(EXPAND(node, problem), fringe)
```

```
function EXPAND( node, problem) returns a set of nodes
    successors ← the empty set
    for each action, result in SUCCESSOR-FN(problem, STATE[node]) do
        s ← a new NODE
        PARENT-NODE[s] ← node;  ACTION[s] ← action;  STATE[s] ← result
        PATH-COST[s] ← PATH-COST[node] + STEP-COST(STATE[node], action, result)
        DEPTH[s] ← DEPTH[node] + 1
        add s to successors
    return successors
```

Search strategies

A strategy is defined by picking the **order of node expansion**

Strategies are evaluated along the following dimensions:

- completeness: does it always find a solution if one exists?
- time complexity: number of nodes generated/expanded
- space complexity: maximum number of nodes in memory
- optimality: does it always find a least-cost solution?

Time and space complexity are measured in terms of

- b : maximum branching factor of the search tree
- d : depth of the least-cost solution
- m : maximum depth of the state space (may be ∞)

Uninformed search strategies

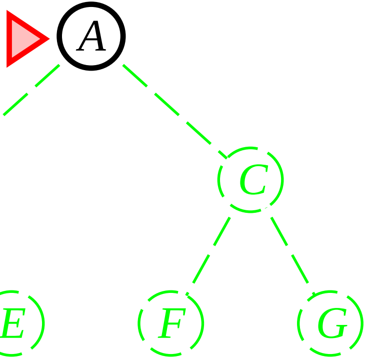
Uninformed strategies use only the information available in the problem definition

- Breadth-first search
- Uniform-cost search
- Depth-first search
- Depth-limited search
- Iterative deepening search

Breadth-first search

Expand shallowest unexpanded node

Implementation: *fringe* is a FIFO queue, i.e., new successors go at end



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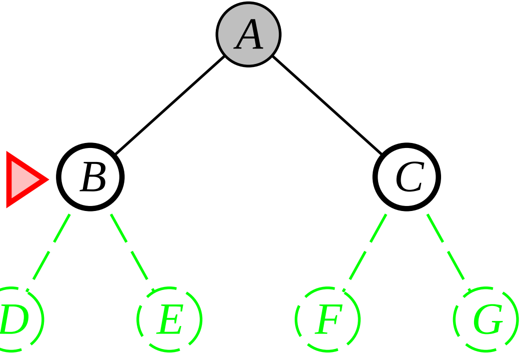

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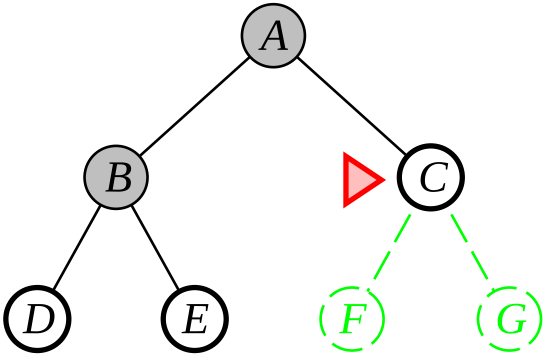

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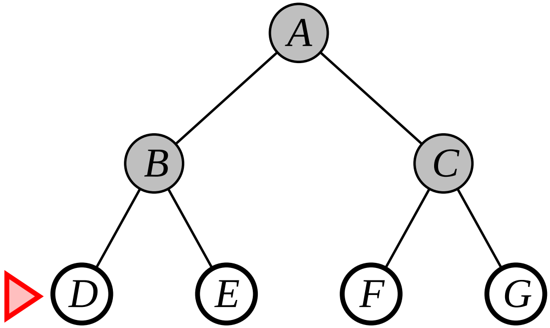

Properties of breadth-first search

- Complete??

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- Optimal?? Yes (if cost = 1 per step); not optimal in general

Space is the big problem; can easily generate nodes at 100MB/sec
so 24hrs = 8640GB.

Uniform-cost search

Expand least-cost unexpanded node

Implementation: *fringe* = queue ordered by path cost, lowest first

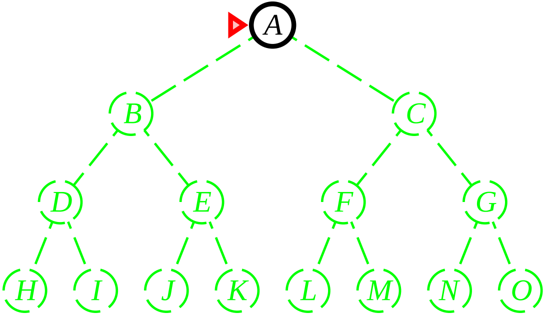
Equivalent to breadth-first if step costs all equal

- Complete?? Yes, if step cost $\geq \epsilon$
- Time?? # of nodes with $g \leq$ cost of optimal solution, $O(b^{\lceil C^*/\epsilon \rceil})$ where C^* is the cost of the optimal solution
- Space?? # of nodes with $g \leq$ cost of optimal solution, $O(b^{\lceil C^*/\epsilon \rceil})$
- Optimal?? Yes—nodes expanded in increasing order of $g(n)$

Depth-first search

Expand deepest unexpanded node

Implementation: *fringe* = LIFO queue, i.e., put successors at front



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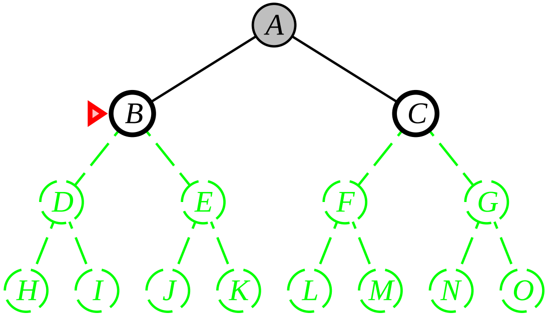

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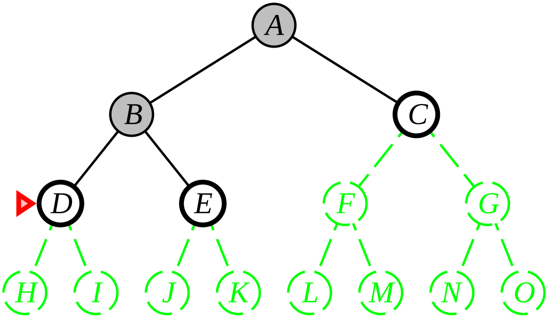

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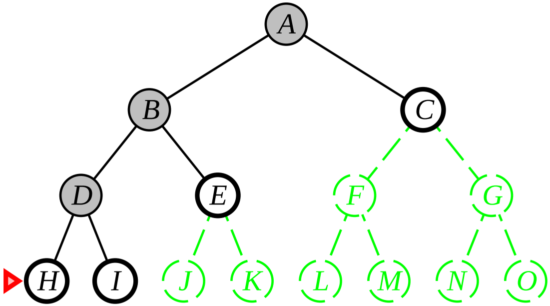

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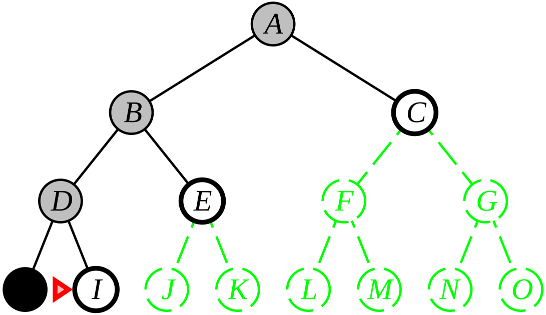

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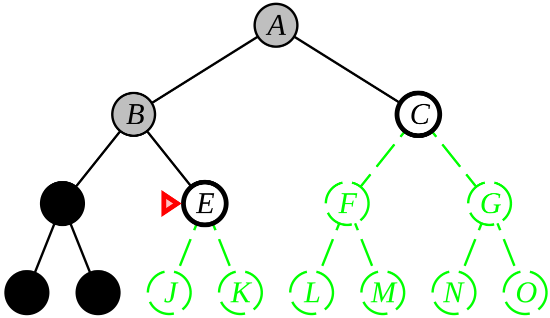

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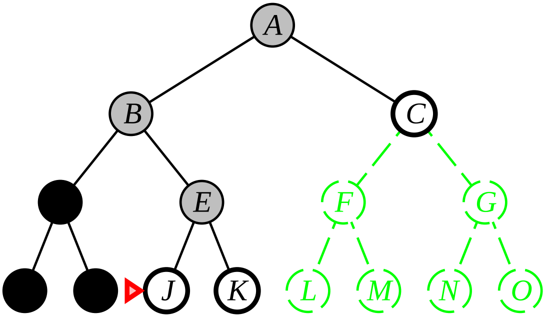

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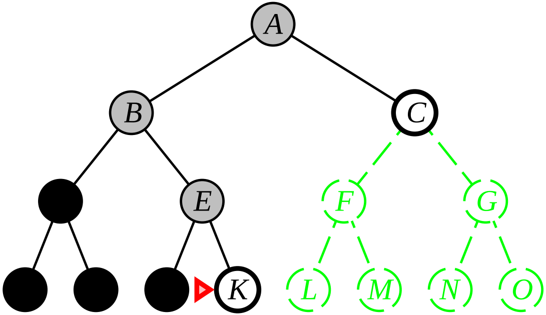

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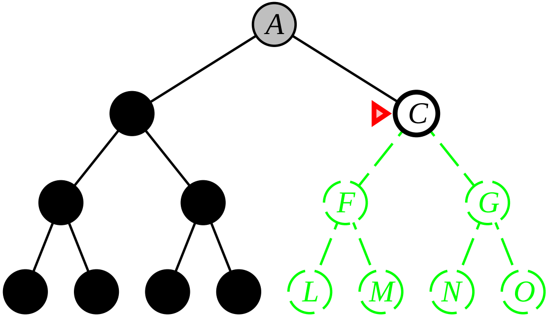

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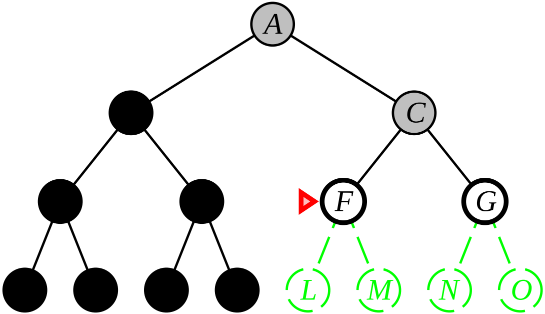

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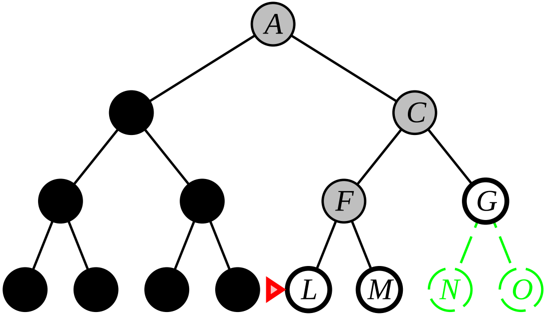

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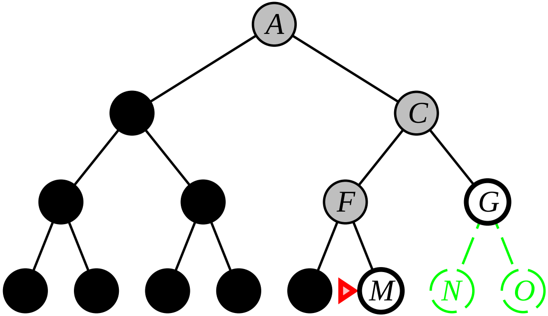

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- Space?? $O(bm)$, i.e., linear space!
- Optimal?? No

Depth-limited search

= depth-first search with depth limit l,
i.e., nodes at depth l have no successors

Recursive implementation:

```
function DEPTH-LIMITED-SEARCH( problem, limit) returns soln/fail/cutoff
    RECURSIVE-DLS(MAKE-NODE(INITIAL-STATE[problem]), problem, limit)

function RECURSIVE-DLS(node, problem, limit) returns soln/fail/cutoff
    cutoff-occurred? ← false
    if GOAL-TEST(problem, STATE[node]) then return node
    else if DEPTH[node] = limit then return cutoff
    else for each successor in EXPAND(node, problem) do
        result ← RECURSIVE-DLS(successor, problem, limit)
        if result = cutoff then cutoff-occurred? ← true
        else if result ≠ failure then return result
    if cutoff-occurred? then return cutoff else return failure
```

Iterative deepening search

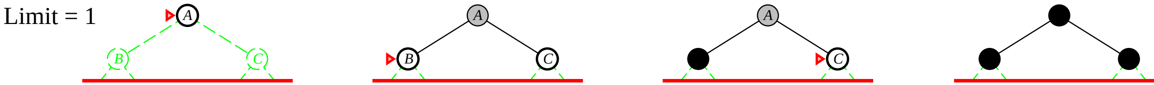
```
function ITERATIVE-DEEPENING-SEARCH( problem) returns a solution
    inputs: problem, a problem

    for depth ← 0 to ∞ do
        result ← DEPTH-LIMITED-SEARCH( problem, depth)
        if result ≠ cutoff then return result
    end
```

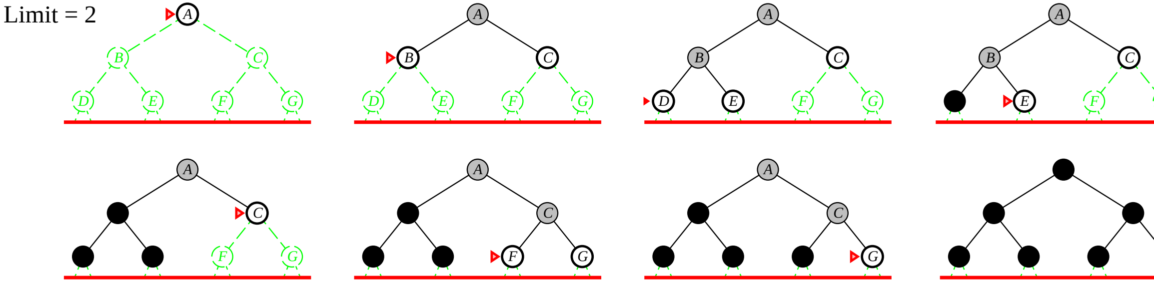
Iterative deepening search $l = 0$

Limit = 0

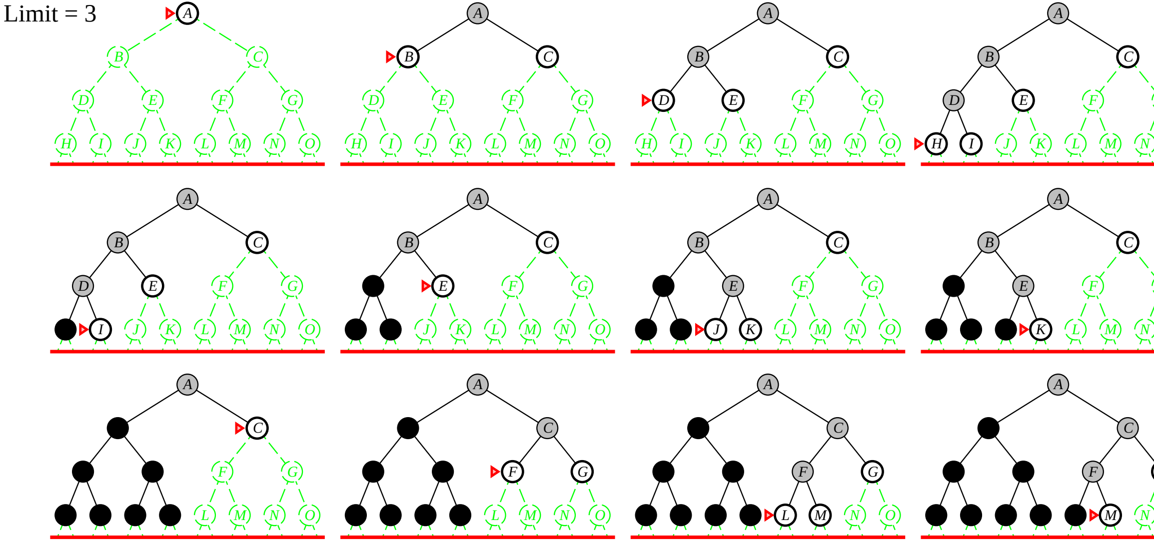
Iterative deepening search $l = 1$



Iterative deepening search $l = 2$



Iterative deepening search $l = 3$



Properties of iterative deepening search

- Complete??

Properties of iterative deepening search

- Complete?? Yes

Properties of iterative deepening search

- Complete?? Yes
- Time??

Properties of iterative deepening search

- Complete?? Yes
- Time?? $(d+1)b^0 + db^1 + (d-1)b^2 + \ldots + b^d = O(b^d)$

Properties of iterative deepening search

- Complete?? Yes
- Time?? $(d+1)b^0 + db^1 + (d-1)b^2 + \ldots + b^d = O(b^d)$
- Space??

Properties of iterative deepening search

- Complete?? Yes
- Time?? $(d+1)b^0 + db^1 + (d-1)b^2 + \ldots + b^d = O(b^d)$
- Space?? $O(bd)$

Properties of iterative deepening search

- Complete?? Yes
- Time?? $(d+1)b^0 + db^1 + (d-1)b^2 + \ldots + b^d = O(b^d)$
- Space?? $O(bd)$
- Optimal??

Properties of iterative deepening search

- Complete?? Yes
- Time?? $(d+1)b^0 + db^1 + (d-1)b^2 + \ldots + b^d = O(b^d)$
- Space?? $O(bd)$
- Optimal?? Yes, if step cost = 1
  Can be modified to explore uniform-cost tree

Numerical comparison for $b = 10$ and $d = 5$, solution at far right leaf:

$$\begin{aligned} N(\text{IDS}) &= 50 + 400 + 3,000 + 20,000 + 100,000 = 123,450 \\ N(\text{BFS}) &= 10 + 100 + 1,000 + 10,000 + 100,000 + 999,990 = 1,111,100 \end{aligned}$$

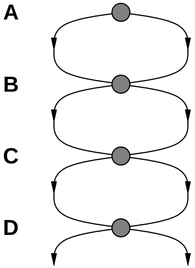
- IDS does better because other nodes at depth d are not expanded
- BFS can be modified to apply goal test when a node is **generated**

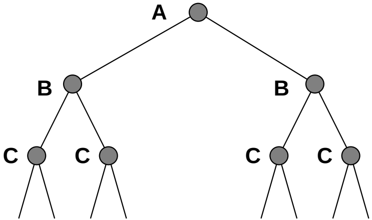
Summary of algorithms

Criterion	Breadth-First	Uniform-Cost	Depth-First	Depth-Limited	Iterative Deepening
Complete?	Yes*	Yes*	No	Yes, if $l \geq d$	Yes
Time	b^{d+1}	$b^{\lceil C^*/\epsilon \rceil}$	b^m	b^l	b^d
Space	b^{d+1}	$b^{\lceil C^*/\epsilon \rceil}$	bm	bl	bd
Optimal?	Yes*	Yes	No	No	Yes*

Repeated states

Failure to detect repeated states can turn a linear problem into an exponential one!





Graph search

```
function GRAPH-SEARCH( problem, fringe) returns a solution, or failure

    closed ← an empty set
    fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
    loop do
        if fringe is empty then return failure
        node ← REMOVE-FRONT(fringe)
        if GOAL-TEST(problem, STATE[node]) then return node
        if STATE[node] is not in closed then
            add STATE[node] to closed
            fringe ← INSERTALL(EXPAND(node, problem), fringe)
    end
```

Summary

- Problem formulation usually requires abstracting away real-world details to define a state space that can feasibly be explored
- Variety of uninformed search strategies
- Iterative deepening search uses only linear space
  and not much more time than other uninformed algorithms
- Graph search can be exponentially more efficient than tree search

Informed Search Algorithms

- Best-first search
- A^* search
- Heuristics

Review: Tree search

```
function TREE-SEARCH( problem, fringe) returns a solution, or failure
    fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
    loop do
        if fringe is empty then return failure
        node ← REMOVE-FRONT(fringe)
        if GOAL-TEST[problem] applied to STATE(node) succeeds return node
        fringe ← INSERTALL(EXPAND(node, problem), fringe)
```

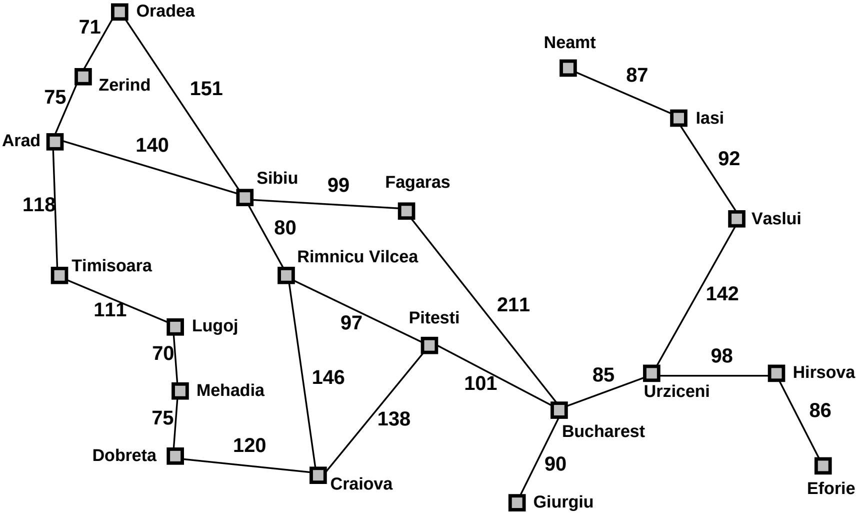
A strategy is defined by picking the **order of node expansion**

Best-first search

- Idea: use an evaluation function for each node
  - estimate of "desirability"

  $\Rightarrow$ Expand most desirable unexpanded node
- Implementation: *fringe* is a queue sorted in decreasing order of desirability
- Special cases:
  - greedy search
  - A^* search

Romania with step costs in km



Straight−line distance
to Bucharest

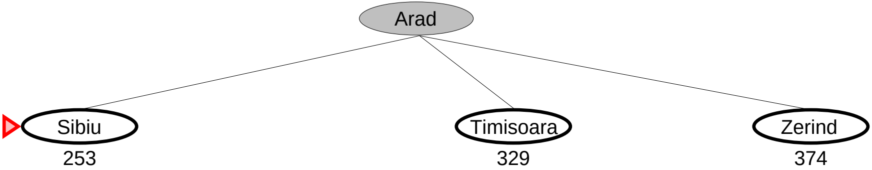
Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	178
Giurgiu	77
Hirsova	151
Iasi	226
Lugoj	244
Mehadia	241
Neamt	234
Oradea	380
Pitesti	98
Rimnicu Vilcea	193
Sibiu	253
Timisoara	329
Urziceni	80
Vaslui	199
Zerind	374

Greedy search

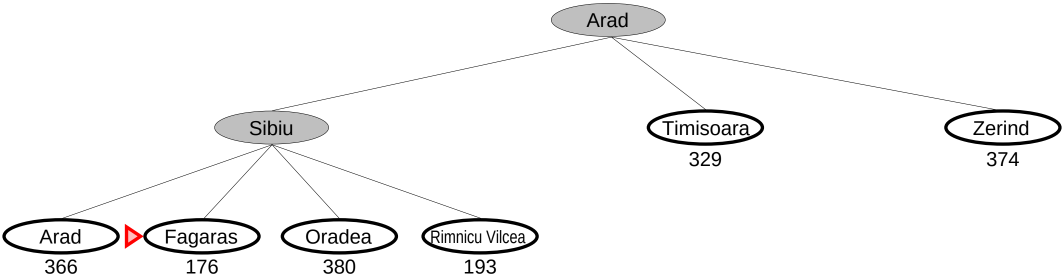
- Evaluation function $h(n)$ (**h**euristic)
  = estimate of cost from n to the closest goal
  - E.g., $h_{\mathrm{SLD}}(n)$ = straight-line distance from n to Bucharest
- Greedy search expands the node that **appears** to be closest to goal

Greedy search example

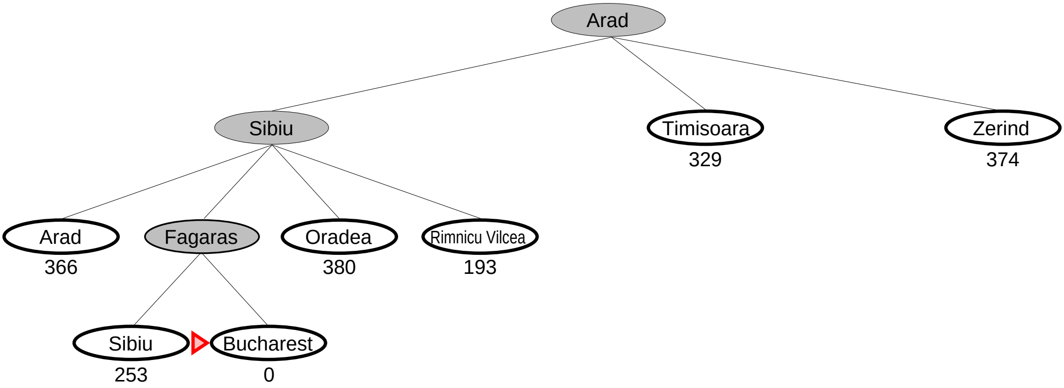
Greedy search example



Greedy search example



Greedy search example



Properties of greedy search

- Complete??

Properties of greedy search

- Complete?? No–can get stuck in loops, e.g.,
  Iasi $\to$ Neamt $\to$ Iasi $\to$ Neamt $\to$
  Complete in finite space with repeated-state checking

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- Optimal??

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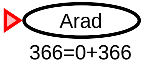
- Space?? $O(b^m)$—keeps all nodes in memory
- Optimal?? No

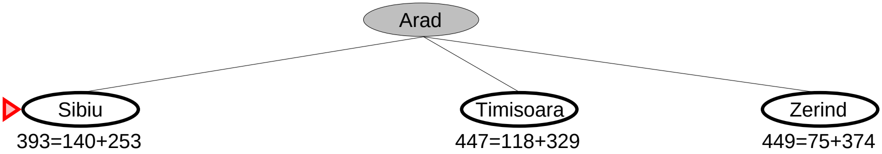
A^* search

- Idea: avoid expanding paths that are already expensive
- Evaluation function $f(n) = g(n) + h(n)$
  - $g(n)$ = cost so far to reach n
    $h(n)$ = estimated cost to goal from n
    $f(n)$ = estimated total cost of path through n to goal
- A^* search uses an admissible heuristic
  i.e., $h(n) \leq h^*(n)$ where $h^*(n)$ is the **true** cost from n.
  (Also require $h(n) \geq 0$, so $h(G) = 0$ for any goal G.)
  - E.g., $h_{\mathrm{SLD}}(n)$ never overestimates the actual road distance
- Theorem: A^* search is optimal

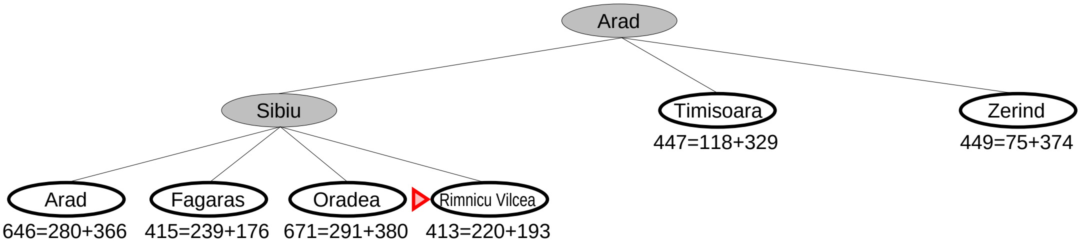
A* search example



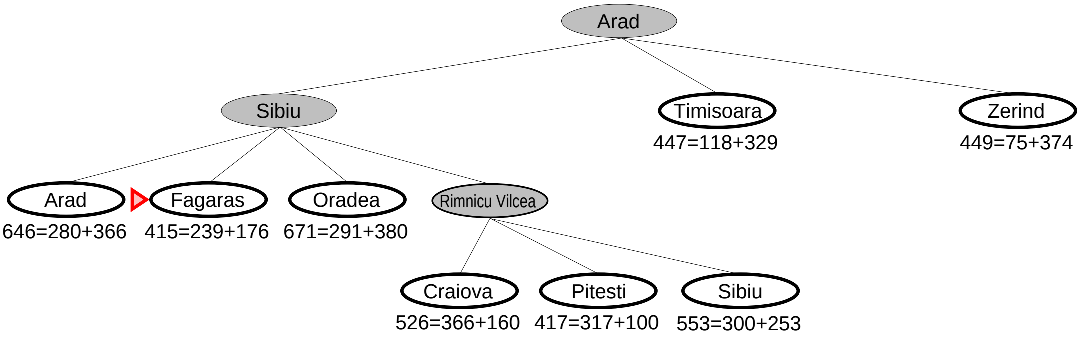
A* search example



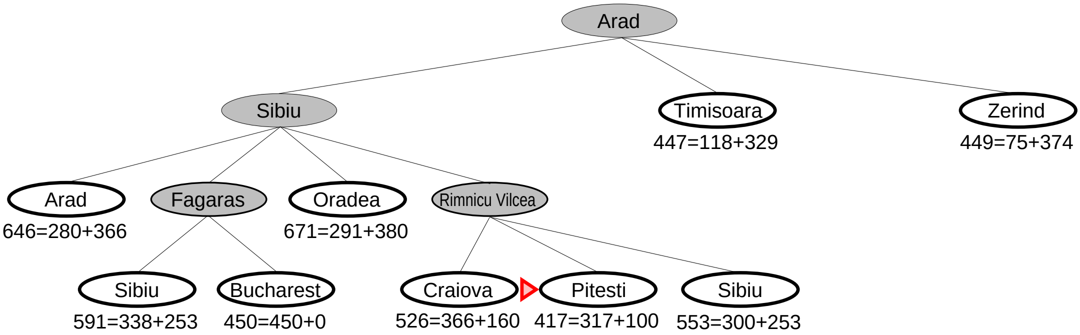
A* search example



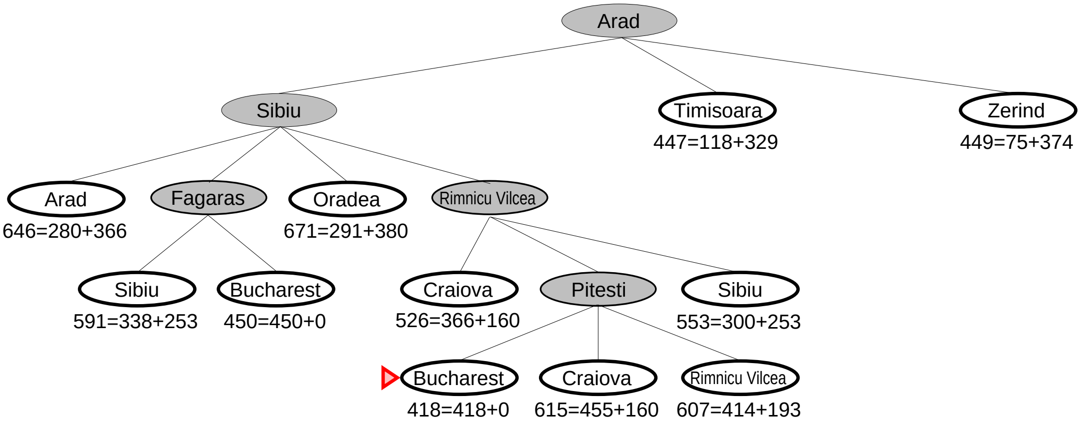
A* search example



A* search example

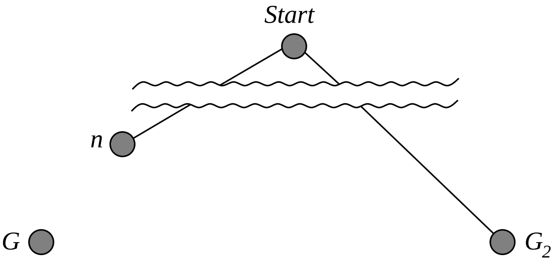


A* search example



Optimality of A* (standard proof)

Suppose some suboptimal goal G_2 has been generated and is in the queue. Let n be an unexpanded node on a shortest path to an optimal goal G_1.



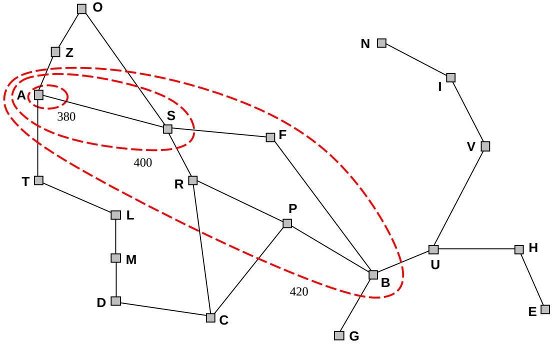
$$
\begin{aligned}
f(G_2) &= g(G_2) && \text{since } h(G_2) = 0 \\
&> g(G_1) && \text{since } G_2 \text{ is suboptimal} \\
&\geq f(n) && \text{since } h \text{ is admissible}
\end{aligned}
$$

Since $f(G_2) > f(n)$, A* will never select G_2 for expansion

Optimality of A* (more useful)

Lemma: A* expands nodes in order of increasing f value*

Gradually adds "f-contours" of nodes (cf. breadth-first adds layers)
Contour i has all nodes with $f = f_i$, where $f_i < f_{i+1}$



Properties of A*

- Complete??
- Time??
- Space??
- Optimal??

Properties of A*

- Complete?? Yes, unless there are infinitely many nodes with $f \leq f(G)$
- Time??
- Space??
- Optimal??

Properties of A*

- Complete?? Yes, unless there are infinitely many nodes with $f \leq f(G)$
- Time?? Exponential in [relative error in h $\times$ length of soln.]
- Space??
- Optimal??

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- Space?? Keeps all nodes in memory
- Optimal??

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- Optimal?? Yes—cannot expand f_{i+1} until f_i is finished

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A* expands all nodes with $f(n) < C^*$

A* expands some nodes with $f(n) = C^*$

A* expands no nodes with $f(n) > C^*$

Proof of lemma: Consistency

A heuristic is consistent if

$$h(n) \leq c(n, a, n') + h(n')$$

If h is consistent, we have

$$\begin{aligned} f(n') &= g(n') + h(n') \\ &= g(n) + c(n, a, n') + h(n') \\ &\geq g(n) + h(n) \\ &= f(n) \end{aligned}$$

I.e., $f(n)$ is nondecreasing along any path.

Admissible heuristics

E.g., for the 8-puzzle:

$h_1(n)$ = number of misplaced tiles
$h_2(n)$ = total Manhattan distance
(i.e., no. of squares from desired location of each tile)

7	2	4
5		6
8	3	1

Start State

1	2	3
4	5	6
7	8	

Goal State

$h_1(S) =??$
$h_2(S) =??$

Admissible heuristics

E.g., for the 8-puzzle:

$h_1(n)$ = number of misplaced tiles
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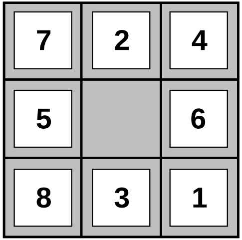
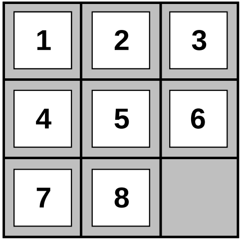


Start State



Goal State

$h_1(S) =??$ 6
$h_2(S) =??$ 4+0+3+3+1+0+2+1 = 14

Dominance

If $h_2(n) \geq h_1(n)$ for all n (both admissible)
then h_2 dominates h_1 and is better for search

Typical search costs:

$d = 14$	IDS = 3,473,941 nodes
	$\text{A}^*(h_1)$ = 539 nodes
	$\text{A}^*(h_2)$ = 113 nodes
$d = 24$	IDS $\approx$ 54,000,000,000 nodes
	$\text{A}^*(h_1)$ = 39,135 nodes
	$\text{A}^*(h_2)$ = 1,641 nodes

Given any admissible heuristics h_a, h_b,

$$h(n) = \max(h_a(n), h_b(n))$$

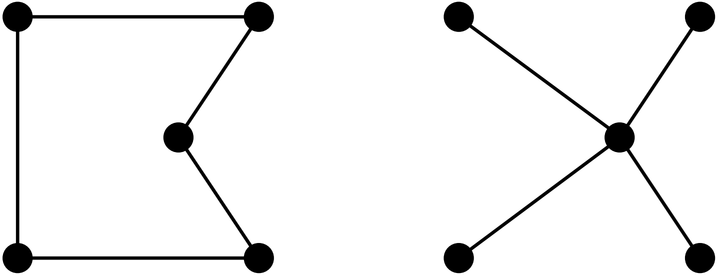
is also admissible and dominates h_a, h_b

Relaxed problems

- Admissible heuristics can be derived from the **exact** solution cost of a **relaxed** version of the problem
- If the rules of the 8-puzzle are relaxed so that a tile can move **anywhere**, then $h_1(n)$ gives the shortest solution
- If the rules are relaxed so that a tile can move to **any adjacent square**, then $h_2(n)$ gives the shortest solution
- Key point: the optimal solution cost of a relaxed problem is no greater than the optimal solution cost of the real problem

Relaxed problems contd.

Well-known example: travelling salesperson problem (TSP)
Find the shortest tour visiting all cities exactly once



Minimum spanning tree can be computed in $O(n^2)$
and is a lower bound on the shortest (open) tour

Summary

- Heuristic functions estimate costs of shortest paths
- Good heuristics can dramatically reduce search cost
- Greedy best-first search expands lowest h
  - incomplete and not always optimal
- A* search expands lowest $g + h$
  - complete and optimal
  - also optimally efficient (up to tie-breaks, for forward search)
- Admissible heuristics can be derived from exact solution of relaxed problems